De Saint-Venant Solution for the Flexure of Cantilevers of Cross-section in the form of Complete and Curtate Circular Sectors, and on the Influence of the Manner of Fixing the Built-in End of the Cantilever on its Deflection.

By MARY SEEGAR, B.Sc. and KARL PEARSON, F.R.S.

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(1) Hitherto, as far as we are aware, the well-known de Saint-Venant solution of the problem of flexure has been limited to a relatively few cases and in all these the cross-section of the beam possessed *biaxial* symmetry, there being an axis of symmetry in the plane of loading and also an axis perpendicular to this plane. De Saint-Venant's view that his solution could be directly applied to asymmetrical sections arises from a misconception. We have recently found it possible to extend the de Saint-Venant solution to certain cases in which there is only one axis of symmetry which may be (1) perpendicular to the plane of loading, or (2) in the plane of loading. The former case has been dealt with by Young, Elderton and Pearson in a memoir published last year.* The flexure in this case is accompanied by torsion, and the problem is not without some bearing on the torsion of aircraft propeller blades, for which, indeed, it was worked out. The latter case for the same sections as were dealt with in that paper was then seen to be solvable, the axis of symmetry being now in and not perpendicular to the plane of loading. As far as we are aware no such cases of flexure with uni-axial symmetry of the cross-section have yet been published, although it is clear that a great variety of sections in actual use fall under this category.

For the particular section we have dealt with in this paper, that of a "trough" or "gutter-pipe" section, represented for mathematical purposes by a curtate circular annulus loaded in its plane of symmetry, we found it needful to introduce a term of the form C log r in addition to the ordinary Fourier solution. We then obtained *two* equations for determining C, one from either curved surface of the annulus, and were checked in the course of our investigation. We are very grateful to Mr. W. M. Macaulay of King's College, Cambridge, for dispelling our difficulty by showing that the two equations could after some reduction be shown to be identical.

* "On the Torsion resulting from Flexure in Prisms with Cross-sections of Uni-axial Symmetry only," 'Drapers' Company Research Memoirs,' Technical Series, No. VII, Cambridge University Press. On pp. 5 and 58 de Saint-Venant's misapprehension is discussed.

Following de Saint-Venant's admirable counsel that the mere algebraic solution of a problem was of little service until the physical meaning of it had been extracted by numerical reduction, we have applied very complicated formulæ to study more exactly than has hitherto been done the influence of the manner of fixing the terminal section of the cantilever on its droop. It is well known that the droop of a cantilever depends upon two factors: (a) the bending-moment deflection as provided by the old Euler-Bernoulli hypothesis, and (b) the shear deflection as developed in the more accurate de Saint-Venant theory. Unfortunately the shear deflection is largely dependent on the manner in which the terminal section of the cantilever is fixed. We have considered what proportion the shear deflection can bear to the bending-moment deflection according to the different methods of fixing which can be adopted within the restricted limits of de Saint-Venant's type of solution,

(2) We suppose a prism or cantilever clamped at one end and loaded perpendicularly to its axis at the other. To fix ideas let the axis be supposed that of z and horizontal, the axis of x be the direction of the load W and vertical, the axis of y be horizontal and perpendicular to the above. The origin is taken on the line of symmetry at a distance $\bar{x}$ from the point where the line of centroids (prism axis) meets the fixed cross-section. The axis of symmetry of each cross-section will thus be the vertical. Let l be the length of the cantilever, I the principal moment of inertia of the cross-section about the horizontal axis in its plane. We will suppose the material isotropic, E being its stretch modulus and η its Poisson's ratio. We shall use the notation of the *History of Elasticity*, according to which the shifts are u, v, w parallel to the axes, the strains s_x, s_y, s_z, σ_{yz}, σ_{zx}, σ_{xy} and the stresses $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, $\widehat{yz}$, $\widehat{zx}$, $\widehat{xy}$. The de Saint-Venant hypothesis makes

$$\widehat{xx} = \widehat{yy} = \widehat{xy} = 0, \tag{i}$$

and leads, as it is easy to show, to:

$$\begin{aligned} u &= \frac{\mathrm{W}}{\mathrm{EI}}\left\{\left(\tfrac{1}{2}\eta\,(x-\bar{x})^2-y^2\right)(l-z)+\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3\right\}+\beta z+\alpha, \\ v &= \frac{\mathrm{W}}{\mathrm{EI}}\,\eta y\,(x-\bar{x})\,(l-z), \\ w &= \chi\,(x,y)-\frac{\mathrm{W}}{\mathrm{EI}}\left\{x-\bar{x})\,(lz-\tfrac{1}{2}z^2)+(x-\bar{x})\,y^2\right\}-\beta''x+\gamma'', \end{aligned} \tag{ii}$$

where $\dfrac{d^2\chi}{dx^2}+\dfrac{d^2\chi}{dy^2}=0$, and β, α, β'', γ'' are constants to be determined by the fixing.

The three stresses which are not zero are given by

$$\widehat{yz} = \mu \left\{ \frac{d\chi}{dy} - \frac{W}{EI}(2+\eta)(x-\bar{x})\, y \right\},$$

$$\widehat{zx} = \mu \left\{ \frac{d\chi}{dx} - \frac{W}{EI}(\tfrac{1}{2}\eta\, \{(x-\bar{x})^2 - y^2\} + y^2) + \beta - \beta'' \right\}.$$

$$\widehat{zz} = -\frac{W}{I}(x-\bar{x})\,(l-z), \tag{iii}$$

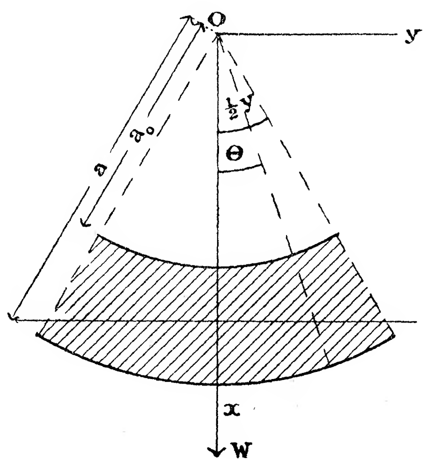
where $\mu = \frac{1}{2}E/(1+\eta)$ is the slide modulus.

Finally, the boundary condition is

$$\widehat{xz}\, dy - \widehat{yz}\, dx = 0. \tag{iv}$$

This is the most general form that can be given on the basis of de Saint-Venant's hypothesis to the solution for the flexure of a prismatic cantilever, when the cross-section is symmetrical with regard to the plane of loading.*

(3) We propose to illustrate this solution in the case of a cantilever of which the cross-section is a curtate circular sector as shown in the accompanying diagram. γ is the sectorial angle; the co-ordinates of a point in the



section are r, θ and the internal and external radii are a_0 and a respectively. The equation for χ will now be

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\chi}{dr}\right) + \frac{d^2\chi}{r^2 d^2\theta} = 0, \tag{v}$$

* See Pearson and Todhunter, 'History of Elasticity,' vol. II, pp. 56-8, and the memoir by Young, Elderton, and Pearson, above referred to, pp. 8–12.

A simple form of the solution of this equation is

$$\chi = \mathrm{C}\log r + \mathrm{S}\left(\mathrm{A}_m r^m + \frac{\mathrm{B}_m}{r^m}\right)\cos m\theta$$
$$+ a_1 r\cos\theta + a_2 r^2\cos 2\theta + a_3 r^3 \cos 3\theta, \tag{vi}$$

where the terms in a_1, a_2, a_3 are introduced to simplify the analysis.

We now turn to the boundary condition

$$\widehat{xz}\,dy - \widehat{yz}\,dx = 0,$$

which provides

$$\frac{d\chi}{dy}dx - \frac{d\chi}{dx}dy = \left[\frac{-\mathrm{W}}{\mathrm{EI}}\{\tfrac{1}{2}\eta\,((z-\bar{x})^2 - y^2) + y^2\} + \beta - \beta''\right]dy$$
$$-\left[-\frac{\mathrm{W}}{\mathrm{EI}}(2+\eta)(x-\bar{x})y\right]dx; \tag{vii}$$

or, transforming to polars,

$$\frac{d\chi}{rd\theta}dr - \frac{d\chi}{dr}rd\theta = dr\left[-\frac{\mathrm{W}}{\mathrm{EI}}\{-r^2(2+\tfrac{1}{2}\eta)\sin\theta + 3r^2\sin^3\theta + 2\bar{x}r\sin\theta\cos\theta\}\right.$$
$$\left. + \left\{-\tfrac{1}{2}\eta\bar{x}^2\frac{\mathrm{W}}{\mathrm{EI}} + \beta - \beta''\right\}\sin\theta\right]$$
$$+ rd\theta\left[-\frac{\mathrm{W}}{\mathrm{EI}}\{r^2(3+\tfrac{1}{2}\eta)\cos\theta - 3r^3\cos^3\theta\right.$$
$$-\bar{x}r(\eta + 2\sin^2\theta)\}$$
$$\left. + \left\{-\tfrac{1}{2}\eta\bar{x}^2\frac{\mathrm{W}}{\mathrm{EI}} + \beta - \beta''\right\}\cos\theta\right]. \tag{viii}$$

Hence, converting the right-hand side into multiple angles, and remembering that either r or θ is constant at the boundary, we have

$$\left(\frac{d\chi}{rd\theta}\right)_{\theta = \pm\frac{1}{2}\gamma} = \left[-\frac{\mathrm{W}}{\mathrm{EI}}\{(\tfrac{1}{4} - \tfrac{1}{2}\eta)r^2\sin\theta - \tfrac{3}{4}r^2\sin 3\theta + \bar{x}r\sin 2\theta\}\right.$$
$$\left. + \left\{-\tfrac{1}{2}\eta\bar{x}^2\frac{\mathrm{W}}{\mathrm{EI}} + \beta - \beta''\right\}\sin\theta\right]_{\theta = \pm\frac{1}{2}\gamma} \text{ for all values of } r, \tag{ix}$$

$$\left(\frac{d\chi}{dr}\right)_{r = \left\{ {a \atop a_0} \right\}} = -\left[\frac{\mathrm{W}}{\mathrm{EI}}\{(\tfrac{3}{4} + \tfrac{1}{2}\eta)r^2\cos\theta - \tfrac{3}{4}r^2\cos 3\theta + \bar{x}r\cos 2\theta - \bar{x}r(1+\eta)\}\right.$$
$$\left. + \left\{-\tfrac{1}{2}\eta\bar{x}^2\frac{\mathrm{W}}{\mathrm{EI}} + \beta - \beta''\right\}\cos\theta\right]_{r = \left\{ {a \atop a_0} \right\}} \text{ for all values of } \theta. \tag{x}$$

Substituting the value of χ from (vi), the first equation gives us for both values of θ:

$$-\mathrm{S}\left(\mathrm{A}_m r^{m-1} + \frac{\mathrm{B}_m}{r^{m+1}}\right)m\sin\tfrac{1}{2}m\gamma - a_1\sin\tfrac{1}{2}\gamma - 2a_2 r\sin\gamma - 3a_3 r^2\sin\tfrac{3}{2}\gamma$$
$$= \left\{-\tfrac{1}{2}\eta\bar{x}^2\frac{\mathrm{W}}{\mathrm{EI}} + \beta - \beta''\right\}\sin\tfrac{1}{2}\gamma - \frac{\mathrm{W}}{\mathrm{EI}}\bar{x}r\sin\gamma$$
$$-r^2\frac{\mathrm{W}}{\mathrm{EI}}\{(\tfrac{1}{4} - \tfrac{1}{2}\eta)\sin\tfrac{1}{2}\gamma - \tfrac{3}{4}\sin\tfrac{3}{2}\gamma\},$$

and this will be satisfied identically if we take

$$\left.\begin{aligned} m &= 2i\pi/\gamma,\ i \text{ being an integer.} \\ a_1 &= \frac{W}{EI}\tfrac{1}{2}\eta\bar{x}^2-(\beta-\beta''), \\ a_2 &= \frac{W}{EI}\tfrac{1}{2}\bar{x}, \\ a_3 &= -\tfrac{1}{3}\frac{W}{EI}\left\{\tfrac{3}{4}-(\tfrac{1}{4}-\tfrac{1}{2}\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\right\}. \end{aligned}\right\} \tag{xi}$$

Thus we have

$$\chi = C\log r+S\left(A_m r^m+\frac{B_m}{r^m}\right)\cos m\theta+\left\{\tfrac{1}{2}\eta\bar{x}^2\frac{W}{EI}-(\beta-\beta'')\right\}r\cos\theta$$

$$+\tfrac{1}{2}\bar{x}\frac{W}{EI}r^2\cos 2\theta-\frac{W}{3EI}\left\{\tfrac{3}{4}-(\tfrac{1}{4}-\tfrac{1}{2}\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\right\}r^3\cos 3\theta, \tag{xii}$$

where $m = 2i\pi/\gamma$.

We now substitute this value of χ in (x) and find

$$S'\left(m\left(A_m a^{m-1}-\frac{B_m}{a^{m+1}}\right)\cos m\theta\right) = -\frac{C'}{a}-\frac{Wa\bar{x}(1+\eta)}{EI}$$

$$+\frac{Wa^2}{EI}\left\{(\tfrac{3}{4}+\tfrac{1}{2}\eta)\cos\theta-(\tfrac{1}{4}-\tfrac{1}{2}\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\cos 3\theta\right\},$$

$$S\left(m\left(A_m a_0^{m-1}-\frac{B_m}{a_0^{m+1}}\right)\cos m\theta\right) = -\frac{C}{a_0}-\frac{Wa_0\bar{x}(1+\eta)}{EI}$$

$$+\frac{Wa_0^2}{EI}\left\{(\tfrac{3}{4}+\tfrac{1}{2}\eta)\cos\theta-(\tfrac{1}{4}-\tfrac{1}{2}\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\cos 3\theta\right\}, \tag{xiii}$$

the range being from $\theta = -\frac{1}{2}\gamma$ to $+\frac{1}{2}\gamma$.

Multiply both these equations by $\cos m\theta$ and integrate throughout the range, we have

$$A_m a^{m-1}-\frac{B_m}{a^{m+1}} = -(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\left\{\frac{3+2\eta}{m(m^2-1)}-\frac{3(1-2\eta)}{m(m^2-9)}\right\}\frac{Wa^2}{EI},$$

$$A_m a_0^{m-1}-\frac{B_m}{a_0^{m-1}} = -(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\left\{\frac{3+2\eta}{m(m^2-1)}-\frac{3(1-2\eta)}{m(m^2-9)}\right\}\frac{Wa_0^2}{EI},$$

leading to

$$A_m = (-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{a^{m+3}-a_0^{m+3}}{a^{2m}-a_0^{2m}}\left\{-\frac{3+2\eta}{m(m^2-1)}+\frac{3(1-2\eta)}{m(m^2-9)}\right\}\frac{W}{EI},$$

$$B_m = -(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{a_0^{m+3}a^{m+3}(a^{m-3}-a_0^{m-3})}{a^{2m}-a_0^{2m}}$$

$$\left\{-\frac{3+2\eta}{m(m^2-1)}+\frac{3(1-2\eta)}{m(m^2-9)}\right\}\frac{W}{EI}. \tag{xiv}$$

It now remains to determine C. Integrate the equations (xiii) between the limits $-\frac{1}{2}\gamma$ and $+\frac{1}{2}\gamma$ of θ, and we find:

$$0 = = \frac{C\gamma}{a} - \frac{Wa\bar{x}(1+\eta)}{EI}\gamma + \frac{Wa^2}{EI}\{(\tfrac{3}{2}+\eta)\sin\tfrac{1}{2}\gamma - \tfrac{1}{3}(\tfrac{1}{2}-\eta)\sin\tfrac{1}{2}\gamma\},$$

$$0 = -\frac{C\gamma}{a} - \frac{Wa_0\bar{x}(1+\eta)}{EI}\gamma + \frac{Wa_0^2}{EI}\{(\tfrac{3}{2}+\eta)\sin\tfrac{1}{2}\gamma - \tfrac{1}{3}(\tfrac{1}{2}-\eta)\sin\tfrac{1}{2}\gamma\}.$$

These equations at first appeared imcompatible, but if $\bar{x} = \frac{4}{3}\,\frac{\sin\frac{1}{2}\gamma}{\gamma}\,\frac{a^3-a_0^3}{a_2-a_0^2}$ be inserted, we reach:

$$C = -\tfrac{4}{3}\,\frac{W(1+\eta)}{EI}\,\frac{a^2a_0^2}{a+a_0}\,\frac{\sin\frac{1}{2}\gamma}{\gamma} \qquad \text{(xv)}$$

from *both*, a value absolutely symmetrical in a and a_0.

Thus we obtain the following value for χ:

$$\chi = \left\{\tfrac{1}{2}\eta\bar{x}^2\frac{W}{EI} - (\beta-\beta'')\right\} r\cos\theta + \tfrac{1}{2}\bar{x}\frac{W}{EI}r^2\cos 2\theta$$

$$-\frac{W}{3\,EI}\cdot\left\{\tfrac{3}{4} - (\tfrac{1}{4}-\tfrac{1}{2}\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\right\} r^3\cos 3\theta - \tfrac{4}{3}\,\frac{W(1+\eta)}{EI}\,\frac{a^2a_0^2}{a+a_0}\,\frac{\sin\frac{1}{2}\gamma}{\gamma}\log_e r$$

$$+\overset{i=0}{\underset{i=1}{S}}(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\left\{-\frac{(3+2\eta)}{m(m^2-1)} + \frac{3(1-2\eta)}{m(m^2-9)}\right\}\frac{W}{EI}\times$$

$$\left\{\frac{a^{m+3}-a_0^{m+3}}{a^{2m}-a_0^{2m}}r^m - \frac{a^{m+3}a_0^{m+3}(a^{m-3}-a_0^{m-3})}{(a^{2m}-a_0^{2m})r^m}\right\}\times\cos m\theta, \qquad \text{(xvi)}$$

where $m = 2i\pi/\gamma$ and $\bar{x} = \frac{4}{3}\,\frac{\sin\frac{1}{2}\gamma}{\gamma}\,\frac{a^3-a_0^3}{a^2-a_0^2}$.

If this value of χ be inserted in equations (ii) and (iii) we have the full solution of the problem of flexure for a prism, the cross-section of which is a curtate circular sector, the load being in its plane of symmetry. It is, as far as we are aware, the first solution for flexure in the case of a section with uniaxial symmetry only, loaded in the direction of that axis. As the paper by Young, Elderton, and Pearson already referred to gives the solution for a prism of the same character loaded perpendicular to its plane of symmetry we have now the full solution for this case whatever be the plane of loading, *i.e.*, the first de Saint-Venant flexure solution has been obtained when the direction of loading has no relation to any plane of symmetry at all.*

The only constants that remain to be determined are the α, β, γ'' and β'' of equation (ii). These depend upon the fixing of the "built-in" terminal section, and they provide the very limited variety of constraint which is

* Of course de Saint-Venant's solutions for rectangle, ellipse, false ellipse, etc., admit of this extension, but being sections of bi-axial symmetry they do not involve either the associated flexual torsion or the presence of the logarithmic terms of this paper.

possible under the mathematical conditions of the de Saint-Venant problem and its solution.

It will be remembered that de Saint-Venant adopted as his fixing conditions (i) that the centroid of the terminal section should be fixed, *i.e.*, $u = v = w = 0$ at that point, and also (ii) the direction of an elementary plane at that point should also be fixed, *i.e.*, $dw/dx = 0$. The conditions of the problem (see equation (ii)) make $dv/dx = 0$ and $du/dy = 0$ at the centroid also. We propose to compare this fixing in its influence on the deflection against that obtained by fixing more widely separated points of the terminal section, as far as such points can be fixed by the de Saint-Venant solution.

(4) We turn next to the value for the deflection. It is needful, however, to settle first what we mean by the "deflection." Turning to the first equation of (ii) we may take the shift at a point x_0 on the axis of symmetry, *i.e.*, $y = 0$, at z; this is

$$u_z = \frac{\mathrm{W}}{\mathrm{EI}}\left\{\tfrac{1}{2}\eta(x_0-\bar{x})(l-z)+\tfrac{1}{2}lz^2-\tfrac{1}{6}z^3\right\}+\beta z+\alpha. \tag{xvii}$$

Now, if f_{x_0} denote the difference between u_l and u_0,

$$f_{x_0} = \tfrac{1}{3}\frac{\mathrm{W}l^3}{\mathrm{EI}}+\beta l-\frac{\eta \mathrm{W}l}{2\,\mathrm{EI}}(x_0-\bar{x}). \tag{xviii}$$

The simplest form, therefore, of the deflection will be obtained if we put $x_0 = \bar{x}$, and then

$$f_{\bar{x}} = \tfrac{1}{3}\frac{\mathrm{W}l^3}{\mathrm{EI}}+\beta l. \tag{xix}$$

In this case $\frac{1}{3}\frac{\mathrm{W}l^3}{\mathrm{EI}}$ is the deflection f_1 of the old Euler-Bernoulli theory, and is due to bending moment, while $\beta l = f_2$ is the deflection added by the de Saint-Venant theory, and is the deflection due to shear. The ratio f_2/f_1 is the ratio of these two deflections, and it is important to ascertain the extent to which this varies with (*a*) the nature of the "fixing" and (*b*) the nature of the cross-section, *i.e.*, in our particular case with the values of a_0, a, and γ. But (xix) is not a good equation for experimental work, because the centroid of the "fixed" terminal will generally be inaccessible, and the centroid of the loaded section if accessible will often, as in trough or tubular sections, not be a physical point of the material of the cantilever itself. For experimental purposes it is almost essential to measure the droops of the cantilever from points of the mid-plane at top or bottom of the material of the cantilever, *i.e.*, where $x_0 = a_0$ or a.

A good practical method is that applied in the memoir already cited:

$$\begin{aligned} u_z &= -\frac{\mathrm{W}l^3}{6\,\mathrm{EI}}\left(\frac{z}{l}\right)^3+\frac{\mathrm{W}l^3}{2\,\mathrm{EI}}\left(\frac{z}{l}\right)^2+\left(\beta l-\frac{\eta\mathrm{W}l}{2\,\mathrm{EI}}(x_0-\bar{x})\right)\frac{z}{l}+\alpha+\frac{\eta\mathrm{W}l}{2\,\mathrm{EI}}(x_0-\bar{x}), \\ &= -a_3\left(\frac{z}{l}\right)^3+a_2\left(\frac{z}{l}\right)^2+a_1\left(\frac{z}{l}\right)+a_0. \end{aligned}$$

Thus $$u_z - u_0 = -a_3\left(\frac{z}{l}\right)^3 + a_2\left(\frac{z}{l}\right)^2 + a_1\left(\frac{z}{l}\right). \tag{xx}$$

Now, measure $u_z - u_0$ for a series of values of z/l, and find a_1, a_2, a_3 by least squares, then $2a_3$ will be the Euler-Bernoulli droop of the cantilever, and a_1 will give the linear term or de Saint-Venant shear term by adding the correction $\frac{1}{2}\frac{\eta W l}{EI}(x_0 - \bar{x})$, which will be known as soon as we have chosen x_0 the point for measurement of our deflections.

The ratio f_2/f_1 will, of course, depend upon the ratio of (length of cantilever to linear dimension of cross-section)^2, and will diminish as the length is increased.

The quantity Wa^2/EI will frequently appear; let us call it q. Let $I = \omega k^2$, where ω is the area of the cross-section, then $W/\omega = L$ is the load per unit of area of cross-section, *i.e.*, the mean shearing stress. k is the radius of gyration about the horizontal axis through the centroid and for the curtate sector, if $a_0/a = \rho$,

$$p = \frac{k^2}{a^2} = \frac{1}{4}(1+\rho^2)\left(1 + \frac{\sin\gamma}{\gamma}\right) - \frac{16}{9}\frac{\sin^2\frac{1}{2}\gamma}{\gamma^2}\left(1 + \frac{\rho^2}{1+\rho}\right)^2. \tag{xxi}$$

Thus $$\frac{f_2}{f_1} = \beta\frac{Wl^2}{3EI} = \frac{\beta}{q}\frac{3a^2}{l^2} = \frac{\beta}{qp}\frac{3k^2}{l^2},$$

where $$q = \frac{Wa^2}{EI} = \frac{L}{Ep} = \frac{\zeta}{p}.$$

Hence, finally, $$\frac{f_2}{f_1} = \frac{\beta}{\zeta}\cdot\frac{3k^2}{l^2}. \tag{xxii}$$

Here $\zeta = L/E$ is independent of the form of the section, and the ratio β/q is what is most easily ascertainable from our formulæ. The value of q does not change with the nature of the fixing, nor the ratios k/l or a/l. Thus the ratio β/q is a good measure of how the shear term increases relative to the bending moment term with changes in the nature of the fixing.

(5) We will consider first the complete sector of sectorial angle γ, and will suppose, in de Saint-Venant's manner, an indefinitely small area fixed at the centroid, G, of the terminal section. Hence we must have $u = v = w = 0$ when $x = \bar{x}$, $y = 0$, $z = 0$. These give us by (ii)

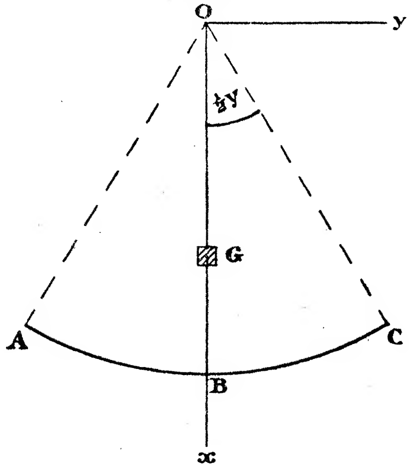
$$\alpha = 0, \qquad \chi(\bar{x}, 0) - \beta''\bar{x} + \gamma'' = 0, \tag{xxiii}$$

and $$dw/dx = 0,$$

or $$0 = (d\chi/dx)_{(\bar{x},0)} - \beta''. \tag{xxiv}$$

Equation (xxiv) will give β. Equation (xxiii) will then determine γ'', while

$\alpha = 0$, and β'' will disappear from the equations. We are, therefore, practically only concerned with the determination of β.



Clearly only the A_m terms will occur in χ, not the B_m terms, and $(d\chi/dx)_{(\bar x, 0)}$ is $(d\chi/dr)_{r=\bar x, \theta=0}$, and accordingly, from (xvi)

$$\beta'' = \tfrac12\eta\left(\frac{\bar x}{a}\right)^2\frac{Wa^2}{EI} - (\beta-\beta'' + \left(\frac{\bar x}{a}\right)^2\frac{Wa^2}{EI} - \frac{Wa^2}{EI}\left\{\tfrac34-(\tfrac14-\tfrac12\eta)\frac{\sin\frac12\gamma}{\sin\frac32\gamma}\right\}\left(\frac{\bar x}{a}\right)^2$$
$$+\sum_{i=1}^{i=\infty}(-1)^i\frac{\sin\frac12\gamma}{\gamma}\left\{-\frac{(3+2\gamma)}{m^2-1}+\frac{3(1-2\eta)}{m^2-9}\right\}\frac{Wa^2}{EI}\left(\frac{\bar x}{a}\right)^{m-1}\Big\},$$

or, remembering that $\bar x = \frac43 a\dfrac{\sin\frac12\gamma}{\gamma}$, $q = \dfrac{Wa^2}{EI}$, we have, re-arranging,

$$\frac{\beta}{q} = \left(\frac{4\sin\frac12\gamma}{3\gamma}\right)^2\left[\tfrac14+\tfrac12\eta+(\tfrac14-\tfrac12\eta)\frac{\sin\frac12\gamma}{\sin\frac32\gamma}\right]$$
$$+\tfrac34\sum_1^{\infty}(-1)^i\left(\frac{4\sin\frac12\gamma}{3\gamma}\right)^m\left[\frac{24-8\eta(m^2-3)}{(m^2-1)(m^2-9)}\right], \qquad \text{(xxv)}$$

where $m = 2\pi i/\gamma$.

The series for β/q is always finite, except when $m = 1$ or 3, *i.e.*, $\gamma = 2\pi$, $\frac23\pi$, and $\frac43\pi$, in which cases we have to proceed to certain limits.

(*a.*) $\gamma = 2\pi$. The terms of the series all vanish, because $\sin\frac12\gamma = 0$, except the first and third. For these terms

$$\text{Limit}_{i\to 1}: \tfrac34\left(\frac{4\sin\frac12\gamma}{3\gamma}\right)^m\left[\frac{24-8\eta(m^2-3)}{(m^2-1)(m^2-9)}\right]$$

has finite values $= -\frac14(3+2\eta)$ and 0. Accordingly

$$\beta/q = 0{\cdot}75+0{\cdot}5\eta. \qquad \text{(xxvi)}$$

(*b.*) $\gamma = \frac23\pi$, or $m = 3i$. Hence the first term of the series becomes infinite, but for this value of γ the term $\sin\frac12\gamma/\sin\frac32\gamma$ also becomes infinite, and the two infinities are approached in the same way, and being of opposite sign, cancel. The method of obtaining the limit was to write $m = 3(1+\epsilon)$, and

expand in powers of ϵ. Since m occurs in a power, a Napierian logarithm term arises in the limit.

$$\text{Limit}_{i\to 1}\left[(-1)^i \tfrac{3}{4}\left(\frac{4\sin\frac{1}{2}\gamma}{3\gamma}\right)^m \left\{\frac{24-8\eta\,(m^2-3)}{(m^2-1)\,(m^2-9)}\right\}+\left(\frac{4\sin\frac{1}{2}\gamma}{3\gamma}\right)^2(\tfrac{1}{4}-\tfrac{1}{2}\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\right]$$

is found to be, since the terms in $1/\epsilon$ cancel:

$$\left(\frac{2\sin\frac{1}{3}\pi}{\pi}\right)^3\left(\frac{11+2\eta}{32}-\tfrac{3}{8}(1-2\eta)\log_e\frac{2\sin\frac{1}{3}\pi}{\pi}\right).$$

Hence

$$\frac{\beta}{q}=\tfrac{1}{4}(1+2\eta)\left(\frac{2\sin\frac{1}{3}\pi}{\pi}\right)^2+2\sum_{i=2}^{i=\infty}(-1)^i\left(\frac{2\sin\frac{1}{3}\pi}{\pi}\right)^{3i}\frac{(1-\eta\,(3i^2-1))}{(i^2-1)\,(3i^2-1)}$$
$$+\left(\frac{2\sin\frac{1}{3}\pi}{\pi}\right)^3\left(\frac{11+2\eta}{32}-\tfrac{3}{8}(1-2\eta)\log_e\frac{2\sin\frac{1}{3}\pi}{\pi}\right). \qquad \text{(xxvii)}$$

(*c*.) $\gamma=4\pi/3$, or $m=3i/2$. The second term of the series now becomes infinite, but so also does the term in $\sin\frac{1}{2}\gamma/\sin\frac{3}{2}\gamma$, and proceeding as in (*b*), we have

$$\text{Limit}_{i\to 2}\left[(-1)^i \tfrac{3}{4}\left(\frac{4\sin\frac{1}{2}\gamma}{3\gamma}\right)^m \left\{\frac{24-8\eta\,(m^2-3)}{(m^2-1)\,(m^2-9)}\right\}+\left(\frac{4\sin\frac{1}{2}\gamma}{3\gamma}\right)^2\frac{1-2\eta}{4}\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\right]$$
$$=\left(\frac{\sin\frac{2}{3}\pi}{\pi}\right)^3\left\{\frac{1-2\eta}{8}\,3\log_e\frac{\sin\frac{2}{3}\pi}{\pi}-\frac{11+2\eta}{32}\right\}.$$

Hence

$$\frac{\beta}{q}=\tfrac{8}{15}(4+\eta)\left(\frac{\sin\frac{2}{3}\pi}{\pi}\right)^{3/2}+8\sum_{i=3}^{i=\infty}(-1)^i\left(\frac{\sin\frac{2}{3}\pi}{\pi}\right)^{\frac{3}{2}i}\left\{\frac{4-\eta\,(3i^2-4)}{(i^2-4)\,(9i^2-4)}\right\}$$
$$+\tfrac{1}{4}(1+2\eta)\left(\frac{\sin\frac{2}{3}\pi}{\pi}\right)^2+\left(\frac{\sin\frac{2}{3}\pi}{\pi}\right)^3\left\{\tfrac{3}{8}(1-2\eta)\log_e\frac{\sin\frac{2}{3}\pi}{\pi}-\frac{11+2\eta}{32}\right\}.$$
(xxviii)

In the limiting case of $\gamma=0$ the series term will be found to vanish and $\beta/q=0{\cdot}148148+0{\cdot}148148\,\eta$.

The following Table for β/q was then calculated from the above series, where $\eta=0{\cdot}25$ gives this ratio for uniconstant isotropy:—

γ.	β/q.	β/q for $\eta=0{\cdot}25$.
°		
0	$0{\cdot}148148+0{\cdot}148148\,\eta$	0·1852
12	$0{\cdot}148153+0{\cdot}146517\,\eta$	0·1848
30	$0{\cdot}148338+0{\cdot}137988\,\eta$	0·1828
45	$0{\cdot}149057+0{\cdot}126944\,\eta$	0·1808
60	$0{\cdot}150718+0{\cdot}115081\,\eta$	0·1795
90	$0{\cdot}157964+0{\cdot}094726\,\eta$	0·1816
120	$0{\cdot}171537+0{\cdot}082112\,\eta$	0·1921
135	$0{\cdot}180999+0{\cdot}079086\,\eta$	0·2008
180	$0{\cdot}221608+0{\cdot}085097\,\eta$	0·2429
225	$0{\cdot}283768+0{\cdot}117675\,\eta$	0·3132
240	$0{\cdot}310197+0{\cdot}135487\,\eta$	0·3441
270	$0{\cdot}373353+0{\cdot}183157\,\eta$	0·4191
315	$0{\cdot}503000+0{\cdot}291850\,\eta$	0·5760
360	$0{\cdot}750000+0{\cdot}250000\,\eta$	0·8125

The values are represented in the upper curve I of the accompanying Diagram I. It will be seen that, while the value falls from $\gamma = 0°$ to a

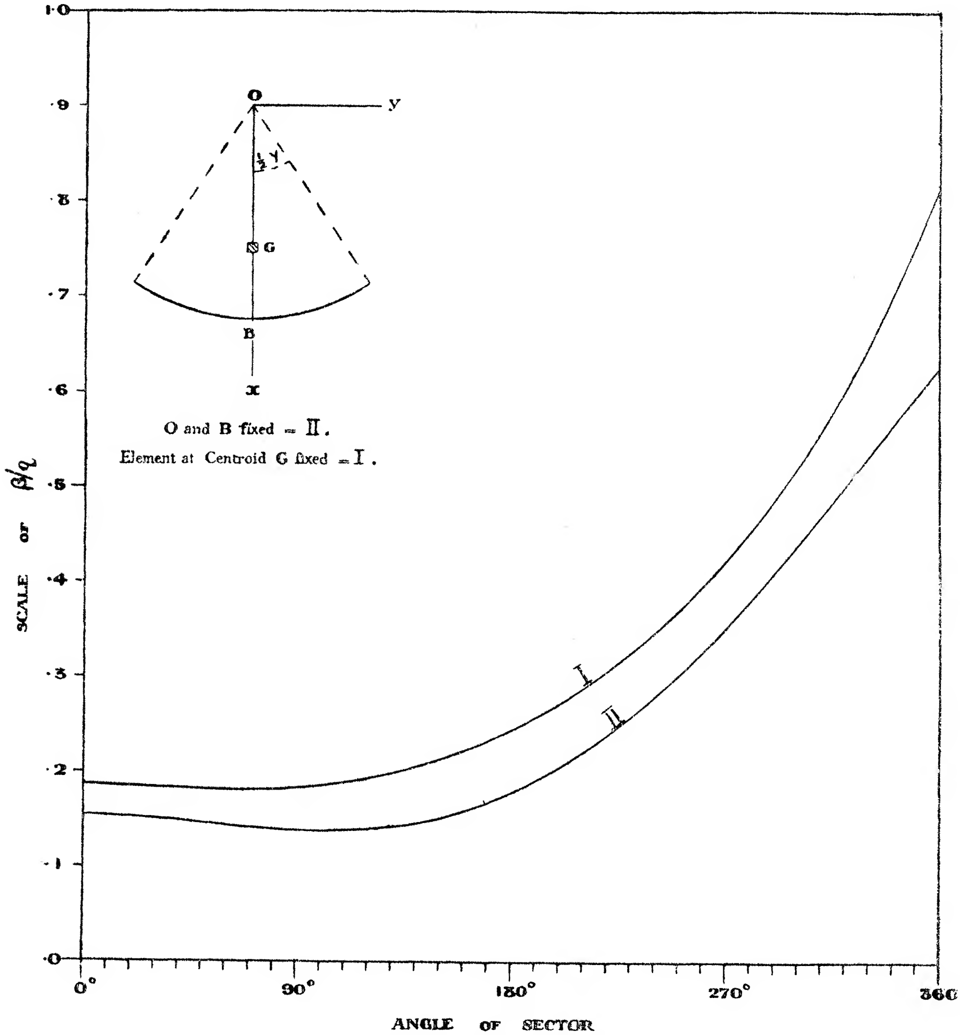


DIAGRAM I.—Non-curtate sectors. Fixing of de Saint-Venant centroidal element (I) and fixing of centre and lowest point of cross-section (II).

minimum at $\gamma = 65°{\cdot}20$, where $\beta/q = 0{\cdot}1794$ the fall is only slight; in fact, we may, for practical purposes, take $\beta/q = 0{\cdot}18$ for uniconstant isotropy for the first quadrant. After $\gamma = 90°$ the value rises continuously and fairly rapidly till it attains its maximum at $\gamma = 360°$, which corresponds to a solid cylinder, with a slit in the plane of loading from periphery to axis. Sectorial prisms, with angles under 12°, may be looked upon as very closely

prisms whose sections are isoceles triangles, and in such cases the deflection due to shear is small compared to that due to bending moment. Thus, for a wedge-shaped cantilever, the length of which is not more than five times its depth, the shear deflection will only be about 2 per cent. of the bending moment deflection. In the case of the slit cylinder ($\gamma = 360°$), however, if the length be but five times the radius, the shear deflection is between 10 per cent. and 11 per cent. of the bending moment deflection; the length must be ten times the radius to reduce this to between 2 per cent. and 3 per cent.

(6) As the method of fixing suggested in the previous section is in practice impossible, we propose to vary it by fixing, as far as possible, the highest and lowest points of the central radius. If we fix the centre O, we have

$$u = v = w = 0 \text{ at } x = y = z = 0.$$

This leads to

$$\tfrac{1}{2}\eta\frac{\mathrm{W}}{\mathrm{EI}}\bar{x}^2 l + \alpha = 0, \quad \text{or} \quad \alpha = -\tfrac{1}{2}\eta\frac{\mathrm{W}}{\mathrm{EI}}\bar{x}^2 l,$$

$$\chi(0, 0) + \gamma'' = 0, \quad \text{or} \quad \gamma'' = -\chi(0, 0) = 0.$$

These equations can accordingly be satisfied by a proper choice of α and γ'', and involve nothing further.

We now try to fix B by taking

$$u = v = w = 0, \text{ when } x = a, \ y = z = 0.$$

This leads to

$$\tfrac{1}{2}\eta\frac{\mathrm{W}}{\mathrm{EI}}(a-\bar{x})^2 l + \alpha = 0,$$

$$\chi(a, 0) + \gamma'' - \beta'' a = 0.$$

The first of these gives us $\alpha = -\frac{1}{2}\eta\frac{\mathrm{W}}{\mathrm{EI}}(a-x)^2 l$, which is incompatible with the previous value for α, unless $\bar{x} = \frac{1}{2}a$. In this case we have

$$\tfrac{1}{2} = \tfrac{4}{3}\frac{\sin\frac{1}{2}\gamma}{\gamma}, \quad \text{or} \quad \tfrac{3}{4} = \frac{\sin\frac{1}{2}\gamma}{\frac{1}{2}\gamma},$$

which gives us $\gamma = 146° \, 11' \, 4''$

as the value of the sectorial angle for which we can fix both O and B. It is this sector which has its centroid at the midpoint of its axis of symmetry.

In the case of any other sector we can fix O, and fix B in the plane OAC, but B must have power of slipping along O B.

Assuming this condition satisfied, then since $\gamma'' = 0$

$$\chi(a, 0) - \beta'' a = 0$$

will settle the value of β. It leads to

$$\frac{\beta}{q}=\sum_{i=1}^{i=\infty}(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\left\{\frac{24-8\eta(m^2-3)}{m(m^2-1)(m^2-9)}\right\}+\tfrac{8}{9}\eta\left(\frac{\sin\frac{1}{2}\gamma}{\gamma}\right)^2+\tfrac{2}{3}\frac{\sin\frac{1}{2}\gamma}{\gamma}-\tfrac{1}{4}$$

$$+\tfrac{1}{12}(1-2\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}, \qquad \text{(xxix)}$$

where $m=2i\pi/\gamma$.

Infinities arise, as in the previous case, for $\gamma=2\pi$, $4\pi/3$, and $2\pi/3$, and we give briefly the results obtained by proceeding to the limits.

(*a.*) $\gamma=2\pi$.

$$\text{Limit}_{i\to 1}(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\,\frac{24-8\eta(m^2-3)}{m(m^2-1)(m^2-9)}=\tfrac{1}{4}(3+2\eta),$$

$$\text{Limit}_{i\to 3}(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\,\frac{24-8\eta(m^2-3)}{m(m^2-1)(m^2-9)}=-\tfrac{1}{36}(1-2\eta).$$

And, accordingly, $$\beta/q=0{\cdot}5+0{\cdot}5\eta. \qquad \text{(xxx)}$$

(*b.*) $\gamma=4\pi/3$. The second term of the series becomes infinite, and also the term in $\sin\frac{1}{2}\gamma/\sin\frac{3}{2}\gamma$.

$$\text{Limit}_{i\to 2}\left[(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\,\frac{24-8\eta(m^2-3)}{m(m^2-1)(m^2-9)}+\tfrac{1}{12}(1-2\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\right]$$

$$=-\sin\tfrac{1}{2}\gamma\frac{(1-2\eta)}{16\pi}-\frac{\sin\frac{1}{2}\gamma}{\gamma}\,\frac{3+2\eta}{24}$$

$$=-\sin\frac{2\pi}{3}\,\frac{1}{32\pi}(5-2\eta)$$

and accordingly,

$$\beta/q=\sum_{i=3}^{i=\infty}\frac{\sin\frac{2}{3}\pi}{\pi}\,\frac{16(4-\eta(3i^2-4))}{3i(9i^2-4)(i^2-4)}+\tfrac{1}{2}\eta\left(\frac{\sin\frac{2}{3}\pi}{\pi}\right)^2$$

$$-\tfrac{1}{4}+\left(\frac{11+2\eta}{32}+\frac{16(4+\eta)}{45}\right)\frac{\sin\frac{2}{3}\pi}{\pi}. \qquad \text{(xxxi)}$$

(*c.*) $\gamma=2\pi/3$. The first term of series becomes infinite.

$$\text{Limit}_{i\to 1}(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\,\frac{24-8\eta(m^2-3)}{m(m^2-1)(m^2-9)}+\tfrac{1}{12}(1-2\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}$$

$$=\frac{5-2\eta}{16\pi}\sin\frac{\pi}{3}.$$

Hence

$$\beta/q=\tfrac{4}{3}\sum_{i=2}^{\infty}(-1)^i\frac{\sin\frac{1}{3}\pi}{\pi}\,\frac{1-\eta(3i^2-1)}{i(9i^2-1)(i^2-1)}+2\eta\left(\frac{\sin\frac{1}{3}\pi}{\pi}\right)^2$$

$$-\tfrac{1}{4}+\frac{21-2\eta}{16}\,\frac{\sin\frac{1}{3}\pi}{\pi} \qquad \text{(xxxii)}$$

From these formulæ the following Table of values for β/q was calculated:

γ.	β/q.	β/q for $\eta = 0\cdot25$.
0°	$\frac{1}{9}+\frac{1}{6}\eta$	0·1528
12	$0\cdot110913+0\cdot164900\,\eta$	0·1521
30	$0\cdot109991+0\cdot158365\,\eta$	0·1496
45	$0\cdot108938+0\cdot149800\,\eta$	0·1464
60	$0\cdot108000+0\cdot139697\,\eta$	0·1429
90	$0\cdot108067+0\cdot118585\,\eta$	0·1377
120	$0\cdot113401+0\cdot101889\,\eta$	0·1389
135	$0\cdot118962+0\cdot097078\,\eta$	0·1432
180	$0\cdot150577+0\cdot104009\,\eta$	0·1766
225	$0\cdot207665+0\cdot150830\,\eta$	0·2454
240	$0\cdot232413+0\cdot175826\,\eta$	0·2764
270	$0\cdot289635+0\cdot239158\,\eta$	0·3494
315	$0\cdot390533+0\cdot360915\,\eta$	0·4808
360	$0\cdot500000+0\cdot500000\,\eta$	0·6250

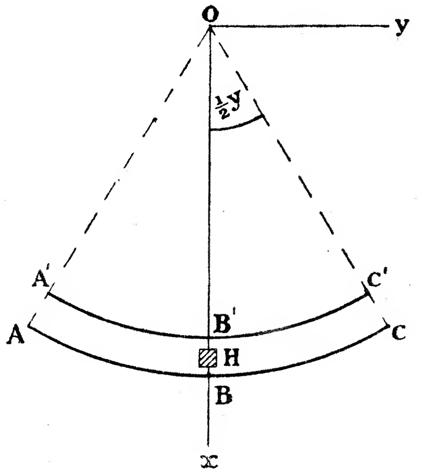
The values are plotted in the lower curve II of Diagram I. It will be observed that for each sectorial angle the proportion of shearing deflection to bending moment deflection is less when we separate as far as possible the fixing points of the terminal section. In other words, de Saint-Venant fixing hypothesis tends to exaggerate the shearing deflection beyond what is likely to arise in any practical fixing, which must tend to fix points at a considerable distance from each other on the terminal section.

(7) We now turn to the annular sector which involves a number of interesting trough sections up to and including the tube with a longitudinal crack. Although the fundamental formulæ can be found for the value of β in special and general cases, the arithmetic is so laborious that we must content ourselves with a definite ratio of internal and external radii. We have taken the thickness, $a-a_0$, of the gutter or trough section to be one-tenth the external radius a or $\rho = a_0/a = 0\cdot9$. The solution is there for any other ratio, but it would require the urgency of greater practical importance to work out a complete table of results corresponding to the double entry, *i.e.*, ρ as well as γ.

Let us first consider the de Saint-Venant method of fixing, *i.e.*, the fixing of a small element at the centroid as far as this is possible. This is theoretically possible as long as the centroid G remains within the area of the cross-section. But when G falls outside the curtate sector—for example, in the case of a semicircular gutter pipe, we could only fix G by attaching the non-material point rigidly by material to the cantilever. But it will then be the method by which this material attachment is made to the terminal section, and not any purely mathematical relations of the non-existent plane element at G which will determine the contribution of

shear to the droop. The values of β/q for the de Saint-Venant fixing would thus be purely mathematical and not practical as soon as the centroid with increasing sectorial angle passed outside the annular sector.* To avoid this difficulty we have modified the de Saint-Venant's fixing, by fixing the elementary area not at the centroid, but at the midpoint, H, of the material in the axis of symmetry. Clearly $\mathrm{OH} = \frac{1}{2}(a+a_0) = \frac{1}{2}a(1+\rho) = 0{\cdot}95\,a$ for our special case.

If the "gutter pipe" has a relatively small thickness the fixing of the highest and lowest points of the actual material, *i.e.*, B and B′, cannot give results differing widely from the fixing of the mid-elementary area, and it does not appear profitable for reasons just stated to fix any purely mathematical point. We have accordingly varied the nature of our fixing to one of a more practically realisable character in the following section. We



have fixed the lowest point B on the external boundary of the cross-section, and the highest points A′, C′ on the internal boundary of the cross-section, as far as this fixing is feasible. The result is of interest, for it indicates that to reduce the droop due to shear the fixing of the maximum number of points on the axis of symmetry is essential. The fixing of laterally situated points is not effective in diminishing shearing droop.

We now turn to the equations for fixing the element at H. We will take $\mathrm{OH} = \frac{1}{2}a(1+\rho) = a\rho'$, so that for our special numerical case $\rho' = \frac{1}{2}(1+\rho) = 0{\cdot}95$.

* The condition for the centroid passing out of the cross-section is

$$\frac{\sin\frac{1}{2}\gamma}{\frac{1}{2}\gamma} = \frac{3}{2}\,\frac{\rho+\rho^2}{1+\rho+\rho^2} \qquad \text{(xxxiii)}$$

or, for our selected value of $\rho = {\cdot}9$, $\dfrac{\sin\frac{1}{2}\gamma}{\frac{1}{2}\gamma} = {\cdot}9464938$, leading to $\gamma = 65^\circ\ 25'\ 8''$.

At H the de Saint-Venant "fixing" gives $u = v = w = du/dx = 0$, and accordingly, from equation (ii),

$$0 = \tfrac{1}{2}\eta \frac{\mathrm{W}a^2 l}{\mathrm{EI}}\left(\rho' - \tfrac{4}{3}\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{1+\rho+\rho^2}{1+\rho}\right)^2 + \alpha,$$

which equation provides the value of α, and

$$0 = \chi(\rho' a, 0) + \gamma'' - \rho' a\beta'',$$

from which equation by (xvi) β'' disappears, and the value of γ'' is provided as soon as β is known.

Lastly:
$$0 = \left(\frac{d\chi}{dr}\right)_{\substack{r=\rho' a\\ \theta=0}} - \beta'',$$

from which equation β'' again disappears, and we have the requisite relation to find β. Writing it out at length, we find:

$$\beta/q = \tfrac{1}{2}\eta\left(\tfrac{4}{3}\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{1+\rho+\rho^2}{1+\rho}\right)^2 + \rho'\tfrac{4}{3}\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{1+\rho+\rho^2}{1+\rho}$$
$$-\rho'^2\left\{\tfrac{3}{4} - \tfrac{1}{4}(1-2\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\right\} - \tfrac{2}{3}(1+\eta)\left(\frac{\rho}{\rho'}\right)^2\frac{\sin\frac{1}{2}\gamma}{\gamma}$$
$$+\sum_1^\infty\left[(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{24-8\eta(m^2-3)}{(m^2-1)(m^2-9)}\frac{\rho'^{2m}-\rho^{2m}+\rho^{m+3}(1-\rho'^{2m})}{\rho'^{m+1}(1-\rho^{2m})}\right]$$

where, as before, $m = 2i\pi/\gamma$. (xxxiv)

As before, infinities arise when γ takes the values 2π, $\frac{2}{3}\pi$, and $\frac{4}{3}\pi$.

(*a*.) $\gamma = 2\pi$. All the terms of the series vanish except $i = 1$, and $i = 3$.

$$\mathrm{Limit}_{i\to 1}\left[(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{24-8\eta(m^2-3)}{(m^2-1)(m^2-9)}\frac{\rho'^{2m}-\rho^{2m}+\rho^{m+3}(1-\rho'^{2m})}{\rho'^{m+1}(1-\rho^{2m})}\right]$$
$$= \tfrac{1}{4}(3+2\eta)\{1+\rho^2-(\rho/\rho')^2\},$$

$$\mathrm{Limit}_{i\to 3}\left[(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{24-8\eta(m^2-3)}{(m^2-1)(m^2-9)}\frac{\rho'^{2m}-\rho^{2m}+\rho^{m+3}(1-\rho'^{2m})}{\rho'^{m+1}(1-\rho^{2m})}\right]$$
$$= -\tfrac{1}{12}(1-2\eta)\rho'^2.$$

Hence
$$\beta/q = \tfrac{1}{4}(3+2\eta)(1+\rho^2-\rho^2/\rho'^2) - \tfrac{3}{4}\rho'^2. \qquad \text{(xxxv)}$$

(*b*.) $\gamma = \frac{2}{3}\pi$. One infinity comes in with the term in $\sin\frac{1}{2}\gamma/\sin\frac{3}{2}\gamma$, and a second with the first term of the series. We have

$$\mathrm{Limit}_{i\to 1}\left[(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{24-8\eta(m^2-3)}{(m^2-1)(m^2-9)}\frac{(\rho'^{2m}-\rho^{2m}+\rho^{m+3}(1-\rho'^{2m})}{\rho'^{m+1}(1-\rho^{2m})} + \rho'^2\tfrac{1}{4}(1-2\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{4}\gamma}\right]$$

$$= \tfrac{3}{4}\rho'^2\frac{\sin\frac{1}{3}\pi}{\pi}\left\{\tfrac{1}{12}(11+2\eta) - (1-2\eta)\left(\log_e\rho' - \frac{1-\rho'^6}{1-\rho^6}\frac{\rho^6}{\rho'^6}\log_e\rho\right\}.$$

And accordingly

$$\beta/q = 2\eta\left[\frac{\sin\frac{1}{3}\pi}{\pi}\left(1+\frac{\rho^2}{2\rho'}\right)\right]^2-\tfrac{3}{4}\rho'^2$$
$$+\frac{\sin\frac{1}{3}\pi}{\pi}\{2\rho'+\rho^2+\tfrac{1}{16}(11+2\eta)\rho'^2-(1+\eta)(\rho/\rho')^2\}$$
$$-\tfrac{3}{4}(1-2\eta)\frac{\sin\frac{1}{3}\pi}{\pi}\rho'^2\left\{\log_e\rho'-\frac{1-\rho'^6}{1-\rho^6}(\rho/\rho')^6\log_e\rho\right\}$$
$$+4\frac{\sin\frac{1}{3}\pi}{\pi}\sum_2^\infty(-1)^i\frac{1-\eta(3i^2-1)}{(i^2-1)(9i^2-1)}\frac{\rho'^{6i}-\rho^{6i}+\rho^{3i+3}(1-\rho'^{6i})}{\rho'^{3i+1}(1-\rho^{6i})}. \tag{xxxvi}$$

(*c*.) $\gamma = 4\pi/3$. One infinity comes in with the term in $\sin\frac{1}{2}\gamma/\sin\frac{3}{2}\gamma$, and a second with the second term of the series. Proceeding to the limit:

$$\text{Limit}_{i\to 2}\left[(-1)^i\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{24-8\eta(m^2-3)}{(m^2-1)(m^2-9)}\frac{\rho'^{2m}-\rho^{2m}+\rho^{m+3}(1-\rho'^{3m})}{\rho'^{m+1}(1-\rho^{2m})}\right.$$
$$\left.+\tfrac{1}{4}\rho'^2(1-2\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\right]$$
$$=\tfrac{3}{8}\rho'^2\frac{\sin\frac{2}{3}\pi}{\pi}\left\{-\tfrac{1}{12}(11+2\eta)+(1-2\eta)\left(\log_e\rho'-\frac{1-\rho'^6}{1-\rho^6}(\rho/\rho')^6\log_e\rho\right)\right\},$$

and accordingly,

$$\beta/q = 2\eta\left[\frac{\sin\frac{2}{3}\pi}{2\pi}\left(1+\frac{\rho^2}{2\rho'}\right)\right]^2-\tfrac{3}{4}\rho'^2+\frac{\sin\frac{2}{3}\pi}{2\pi}[2\rho'+\rho^2-\tfrac{1}{16}(11+2\eta)\rho'^2$$
$$-(1+\eta)(\rho/\rho')^2]$$
$$+\tfrac{3}{4}(1-2\eta)\frac{\sin\frac{2}{3}\pi}{2\pi}\rho'^2\left\{\log_e\rho'-\frac{1-\rho'^6}{1-\rho^2}(\rho/\rho')^6\log_e\rho\right\}$$
$$+\tfrac{16}{15}\frac{\sin\frac{2}{3}\pi}{2\pi}(4+\eta)\frac{\rho'^3(1+\rho\tfrac{3}{2})-\rho^3(1-\rho'^3)}{\rho'^{5/2}(1+\rho^{3/2})}$$
$$+16\frac{\sin(2\pi/3)}{2\pi}\sum_3^\infty\left[(-1)^i\frac{4-\eta(3i^2-4)}{(i^2-4)(9i^2-4)}\frac{\rho'^{3i}-\rho^{3i}+\rho^{\frac{3}{2}i+3}(1-\rho'^{3i})}{\rho'^{\frac{3}{2}i+1}(1-\rho^{3i})}\right]. \tag{xxxvii}$$

From formulæ (xxxiv)–(xxxvii) the following Table for β/q has been constructed. It indicates that the droop due to shear is greatest for the complete tube with a longitudinal slit along the top, but is never very significant, even for this case, amounting only to about 1·5 per cent., when the length of the tube is only five times its external radius. For semi-circular and flatter "gutter pipe" sections the shearing droop is negligible. This is, of course, on the assumption that a small element at the mid-point of the thickness in the median plane has been fixed in the de Saint-Venant manner. In the present illustration such a fixing can hardly differ practically from clamping the mid-thickness in a vice. The values of β/q are plotted as (I) in the accompanying Diagram II:—

γ.	β/q.	β/q for $\eta = 0 \cdot 25$.
°		
0	$0 \cdot 002498 + 0 \cdot 002498\,\eta$	$0 \cdot 0031$
12	$0 \cdot 002501 + 0 \cdot 000704\,\eta$	$0 \cdot 0027$
30	$0 \cdot 002546 + 0 \cdot 000108\,\eta$	$0 \cdot 0026$
45	$0 \cdot 002616 + 0 \cdot 000355\,\eta$	$0 \cdot 0027$
60	$0 \cdot 002715 + 0 \cdot 001089\,\eta$	$0 \cdot 0030$
90	$0 \cdot 002988 + 0 \cdot 004879\,\eta$	$0 \cdot 0042$
120	$0 \cdot 003349 + 0 \cdot 014288\,\eta$	$0 \cdot 0069$
135	$0 \cdot 003619 + 0 \cdot 021830\,\eta$	$0 \cdot 0090$
180	$0 \cdot 004308 + 0 \cdot 061219\,\eta$	$0 \cdot 0196$
225	$0 \cdot 005141 + 0 \cdot 129299\,\eta$	$0 \cdot 0375$
240	$0 \cdot 005526 + 0 \cdot 157970\,\eta$	$0 \cdot 0450$
270	$0 \cdot 006011 + 0 \cdot 224066\,\eta$	$0 \cdot 0620$
315	$0 \cdot 006798 + 0 \cdot 338411\,\eta$	$0 \cdot 0914$
360	$0 \cdot 007495 + 0 \cdot 456247\,\eta$	$0 \cdot 1216$

It will be seen that, up to the semicircular trough, there is hardly any droop due to shear.

(8) We now turn to our last illustration, that of the "gutter pipe" with the lowest point of the external surface and the highest points of the internal surface fixed as far as is feasible. The complete fixing of these points is not within the reach of the mathematical solution. We shall realise the solution by supposing the three points represented by pins parallel to the generators of the tube or portion of a tube, each carrying a nut. A rigid vertical plate has now cut in it a vertical and a horizontal slot. The two pins of the internal boundary work in the horizontal, the pin of the external surface in the vertical slot. In other words, the point on the mid-line of the external surface must have vertical play, and the extreme points on the internal surface need horizontal play. The mathematical conditions are accordingly

$$w = v = 0, \quad \text{when} \quad r = a, \quad \theta = 0.$$

$$u = w = 0, \quad \text{when} \quad r = a_0, \quad \theta = \pm\tfrac{1}{2}\gamma.$$

These conditions lead to

$$0 = \frac{\mathrm{W}a^2}{\mathrm{EI}}\left\{\tfrac{1}{2}\eta\left(\rho\cos\tfrac{1}{2}\gamma - \tfrac{4}{3}\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{1+\rho+\rho^2}{1+\rho}\right)^2 - \rho^2\sin^2\tfrac{1}{2}\gamma\right\} + \alpha,$$

which determines α. Further:

$$0 = \chi(a_0, \pm\tfrac{1}{2}\gamma) - \frac{\mathrm{W}a^3}{\mathrm{EI}}\left(\rho\cos\tfrac{1}{2}\gamma - \tfrac{4}{3}\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{1+\rho+\rho^2}{1+\rho}\right)\rho^2\sin^2\tfrac{1}{2}\gamma$$
$$-\beta'' a_0\cos\tfrac{1}{2}\gamma + \gamma'',$$

$$0 = \chi(a, 0) - \beta'' a + \gamma'',$$

which last two equations suffice to determine γ'' and β, the first of the two

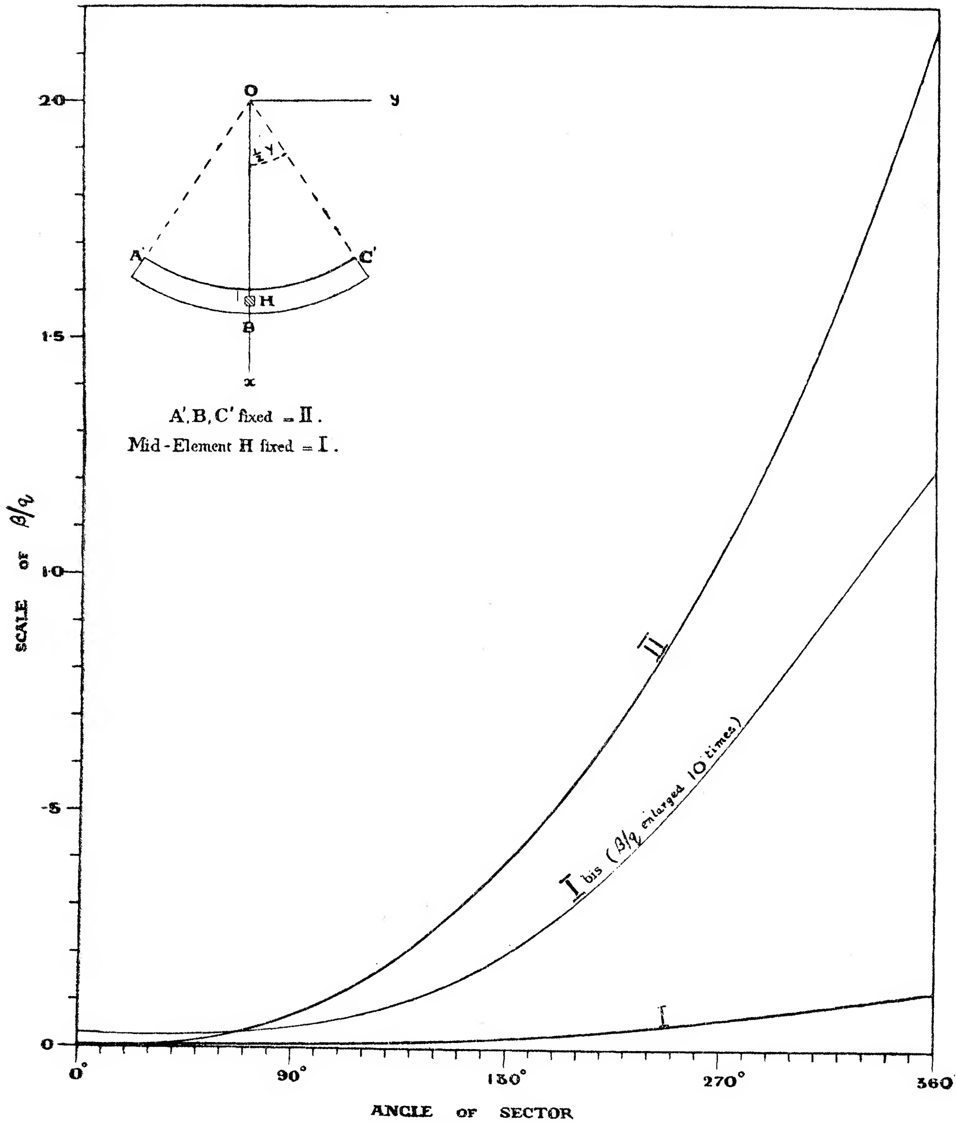


DIAGRAM II.—Curtate sectors. Fixing of a central element in de Saint-Venant's fashion (I) and fixing of lowest and two highest points (II).

being the same for either $\pm\frac{1}{2}\gamma$. Subtracting to get of γ'', we find, after considerable reductions, that

$$\beta/\gamma = \tfrac{1}{2}\eta\left(\tfrac{4}{3}\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{1+\rho+\rho^2}{1+\rho}\right)^2+\tfrac{2}{3}\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{1-\rho^3}{1-\rho\cos\frac{1}{2}\gamma}-\tfrac{1}{4}\frac{1-\rho^3\cos\frac{1}{2}\gamma}{1-\rho\cos\frac{1}{2}\gamma}$$

$$+\tfrac{1}{12}(1-2\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\frac{1-\rho^3\cos\frac{3}{2}\gamma}{1-\rho\cos\frac{1}{2}\gamma}$$

$$+\tfrac{4}{3}(1+\eta)\frac{\sin\frac{1}{2}\gamma}{\gamma}\frac{\rho^2}{(1+\rho)(1-\rho\cos\frac{1}{2}\gamma)}\log_e\rho$$

$$-\frac{\sin\frac{1}{2}\gamma}{\gamma(1-\rho\cos\frac{1}{2}\gamma)}\left[\sum_{i=1,3,5\ldots}^{\infty}(1-\rho^3)\frac{1+\rho^m}{1-\rho^m}\left\{\frac{24-8\eta(m^2-3)}{m(m^2-1)(m^2-9)}\right\}\right.$$

$$\left.-\sum_{i=2,4,6\ldots}^{\infty}(1+\rho^3)\frac{1-\rho^m}{1+\rho^m}\left\{\frac{24-8\eta(m^2-3)}{m(m^2-1)(m^2-9)}\right\}\right], \qquad \text{(xxxviii)}$$

where $m = 2i\pi/\gamma$.

The infinities occur as in previous cases when $\gamma = 2\pi$, $\frac{2}{3}\pi$, or $\frac{4}{3}\pi$. Since $1-\rho\cos\frac{1}{2}\gamma$ is always positive and greater than zero, this provides no infinity.

(*a.*) $\gamma = 2\pi$. All the terms of the series vanish, owing to the factor $\sin\frac{1}{2}\gamma$, except the first and the third.

$$\text{Limit}_{i\to 1}\left[\frac{-\sin\frac{1}{2}\gamma}{\gamma(1-\rho\cos\frac{1}{2}\gamma)}(1-\rho^3)\frac{(1+\rho^m)}{(1-\rho^m)}\frac{24-8\eta(m^2-3)}{m(m^2-1)(m^2-9)}\right],$$

$$= \tfrac{1}{4}(3+2\eta)(1+\rho+\rho^2),$$

$$\text{Limit}_{i\to 3}\left[\frac{-\sin\frac{1}{2}\gamma}{\gamma(1-\rho\cos\frac{1}{2}\gamma)}(1-\rho^3)\frac{1+\rho^m}{1-\rho^m}\frac{24-8\eta(m^2-3)}{m(m^2-1)(m^2-9)}\right]$$

$$= -\tfrac{1}{36}(1-2\eta)(1-\rho-\rho^2).$$

Hence $$\beta/q = \tfrac{1}{4}(3+2\eta)(1+\rho+\rho^2)-\tfrac{1}{4}(1-\rho+\rho^2). \qquad \text{(xxxix)}$$

(*b.*) $\gamma = \frac{2}{3}\pi$. The first or $i = 1$ term becomes infinite, and we must proceed to the limit of this term in conjunction with the term in $1/\sin\frac{3}{2}\gamma$.

We have:

$$\text{Limit}_{i\to 1}\left[\tfrac{1}{12}(1-2\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\frac{1-\rho^3\cos\frac{3}{2}\gamma}{1-\rho\cos\frac{1}{2}\gamma}\right.$$

$$\left.-\frac{\sin\frac{1}{2}\gamma}{\gamma(1-\rho\cos\frac{1}{2}\gamma)}(1-\rho^3)\frac{1+\rho^m}{1-\rho^m}\frac{24-8\eta(m^2-3)}{m(m^2-1)(m^2-9)}\right]$$

$$= \frac{1+\rho^3}{1-\frac{1}{2}\rho}\frac{\sin\frac{1}{3}\pi}{\pi}\left[\tfrac{1}{48}(15-6\eta)-\tfrac{1}{2}(1-2\eta)\frac{\rho^3\log_e\rho}{1-\rho^6}\right].$$

Hence:

$$\beta/q = 2\eta\left(\frac{\sin\frac{1}{3}\pi}{\pi}\frac{(1+\rho+\rho^2)}{1+\rho}\right)^2 - \frac{1}{4}\frac{1-\frac{1}{2}\rho^3}{1-\frac{1}{2}\rho}$$

$$+\frac{\sin\frac{1}{3}\pi}{\pi}\left(\frac{1-\rho^3}{1-\frac{1}{2}\rho}+\frac{1}{16}(5-2\eta)\frac{1+\rho^3}{1-\frac{1}{2}\rho}\right)$$

$$+\frac{\sin\frac{1}{3}\pi}{\pi}\frac{\log_e\rho}{1-\frac{1}{2}\rho}\left(2(1+\eta)\frac{\rho^2}{1+\rho}-\frac{1}{2}(1-2\eta)\frac{\rho^3}{1-\rho^3}\right)$$

$$-\frac{4}{3}\frac{\sin\frac{1}{3}\pi}{\pi}\frac{1}{1-\frac{1}{2}\rho}\left[\sum_{i=3,5,7\ldots}^{\infty}(1-\rho^3)\frac{1+\rho^{3i}}{1-\rho^{3i}}\cdot\frac{1-\eta(3i^2-1)}{i(i^2-1)(9i^2-1)}\right.$$

$$\left.-\sum_{i=2,4,6\ldots}^{\infty}(1+\rho^3)\frac{1-\rho^{3i}}{1+\rho^{3i}}\cdot\frac{1-\eta(3i^2-1)}{i(i^2-1)(9i^2-1)}\right]. \tag{xl}$$

(*c*.) $\gamma = 4\pi/3$. The second or $i = 2$ term becomes infinite in (xxxviii), but is balanced by the infinity of the fourth term. We find:

$$\text{Limit}_{i\to 2}\left[\frac{1}{12}(1-2\eta)\frac{\sin\frac{1}{2}\gamma}{\sin\frac{3}{2}\gamma}\frac{1-\rho^3\cos\frac{3}{2}\gamma}{1-\rho\cos\frac{1}{2}\gamma}\right.$$

$$\left.+\frac{\sin\frac{1}{2}\gamma}{\gamma(1-\rho\cos\frac{1}{2}\gamma)}(1+\rho^3)\frac{1-\rho^m}{1+\rho^m}\frac{24-\eta(m^2-3)}{m(m^2-1)(m^2-9)}\right]$$

$$= -\frac{1-\rho^3}{1+\frac{1}{2}\rho}\frac{\sin\frac{2}{3}\pi}{2\pi}\left[\frac{15-6\eta}{48}+\frac{1}{2}(1-2\eta)\frac{\rho^3\log_e\rho}{1-\rho^5}\right].$$

Hence:

$$\beta/q = 2\eta\left(\frac{\sin\frac{2}{3}\pi}{2\pi}\frac{1+\rho+\rho^2}{1+\rho}\right)^2 - \frac{1}{4}\frac{1+\frac{1}{2}\rho^3}{1+\frac{1}{2}\rho}$$

$$+\frac{\sin\frac{2}{3}\pi}{2\pi}\left(\frac{1-\rho^3}{1+\frac{1}{2}\rho}-\frac{1}{16}(5-2\eta)\frac{1-\rho^3}{1+\frac{1}{2}\rho}\right)$$

$$+\frac{\sin\frac{2}{3}\pi}{2\pi}\frac{\log_e\rho}{1+\frac{1}{2}\rho}\left(2(1+\eta)\frac{\rho^2}{1+\rho}-\frac{1}{2}(1-2\eta)\frac{\rho^3}{1+\rho^3}\right)$$

$$-\frac{32}{3}\frac{\sin\frac{2}{3}\pi}{2\pi}\frac{1}{1+\frac{1}{2}\rho}\left[\sum_{i=1,3,5\ldots}^{\infty}(1-\rho^3)\frac{1+\rho^{\frac{3}{2}i}}{1-\rho^{\frac{3}{2}i}}\frac{4-\eta(3i^2-4)}{i(i^2-4)(9i^2-4)}\right.$$

$$\left.-\sum_{i=4,6,8\ldots}^{\infty}(1+\rho^3)\frac{1-\rho^{\frac{3}{2}i}}{1+\rho^{\frac{3}{2}i}}\frac{4-\eta(3i^2-4)}{i(^2-4)(9i^2-4)}\right]. \tag{xli}$$

From formula (xxxviii) to (xli) the following Table for β/q has been computed. It indicates straight off that the fixing of points off the axis of symmetry of the cross-section is far less effective than fixing points on the axis of symmetry in reducing the droop due to shear. The values of β/q are plotted as II in Diagram II:—

γ.	βq.	β/q for $\eta = 0\cdot25$.
0°	$0\cdot001665 + 0\cdot002082\,\eta$	$0\cdot0022$
12	$0\cdot001666 + 0\cdot000933\,\eta$	$0\cdot0019$
30	$0\cdot003807 - 0\cdot005679\,\eta$	$0\cdot0024$
45	$0\cdot010922 - 0\cdot012605\,\eta$	$0\cdot0078$
60	$0\cdot024593 - 0\cdot017785\,\eta$	$0\cdot0201$
90	$0\cdot072108 - 0\cdot018782\,\eta$	$0\cdot0674$
120	$0\cdot145750 - 0\cdot028483\,\eta$	$0\cdot1386$
135	$0\cdot186336 + 0\cdot009818\,\eta$	$0\cdot1888$
180	$0\cdot368036 + 0\cdot091291\,\eta$	$0\cdot3909$
225	$0\cdot606700 + 0\cdot247616\,\eta$	$0\cdot6686$
240	$0\cdot701470 + 0\cdot320060\,\eta$	$0\cdot7815$
270	$0\cdot915986 + 0\cdot499232\,\eta$	$1\cdot0408$
315	$1\cdot308780 + 0\cdot863110\,\eta$	$1\cdot5242$
360	$1\cdot805000 + 1\cdot355000\,\eta$	$2\cdot1438$

The droop due to shear is clearly of no importance until the sectorial angle reaches 90°. Then it rapidly increases, until β/q becomes about 18 times as large as for the element-fixing when we deal with a split tube.

While in the case of the complete sector we have seen that there is no substantial difference in the shearing droop between the cases of fixing a centroidal element (de Saint-Venant's method) and fixing two points on the mid-section, we see that very considerable differences arise in the case of the annular or trough section according as we fix an element on the mid-plane, or a point on the bottom and two at the top edges. The physical reasons for this great difference in shearing droop are not immediately obvious. Experimental testing of these two methods of fixing would be fairly easy, and at the same time interesting results might be obtained by fixing on the terminal a larger number of points than are compatible with de Saint-Venant's mathematical solution.

The ideal mathematical solution ought to admit any shearing stress over the free end of the cantilever, no stress over the generators of the prism, and any shifts (including the system $u = v = w = 0$) over the whole of the fixed end. But, so far, such very general conditions have defied mathematical treatment. Indeed, as far as we are aware, no solution of any importance has yet been obtained for an elastic body when it is subject to given shifts over one portion and given stresses over the remainder of its surface.